Numerical Computational Techniques

Unit - III

Numerical Differentiation and Integration

* Numerical differentiation using Newton's Forward and backward differences interpolation methods (equal intervals)

* Numerical integration by
  - Trapezoidal rule
  - Simpson's $\frac{1}{3}^{rd}$ rule

Double integration using
  - Trapezoidal rule
  - Simpson's rules.

Newtons Forward difference formula for derivatives

For non-tabular values.

$$\frac{dy}{dx} = \frac{1}{h}\left[\Delta y_0 + \left(\frac{2p-1}{2!}\right)\Delta^2 y_0 + \left(\frac{3p^2-6p+2}{6}\right)\Delta^3 y_0 + \left(\frac{2p^3-9p^2+11p-3}{12}\right)\Delta^4 y_0 + \cdots\right]$$

$$\frac{d^2y}{dx^2} = \frac{1}{h^2}\left[\Delta^2 y_0 + (p-1)\Delta^3 y_0 + \frac{6p^2-18p+11}{12}\Delta^4 y_0 + \cdots\right]$$

$$\frac{d^3y}{dx^3} = \frac{1}{h^3}\left[\Delta^3 y_0 + \frac{12p-18}{12}\Delta^4 y_0 + \cdots\right]$$

where $p = \frac{x-x_0}{h}$

For tabular values. at $x = x_0$

$$\left[\frac{dy}{dx}\right]_{\text{at } x=x_0} = \frac{1}{h}\left[\Delta y_0 - \frac{\Delta^2 y_0}{2} + \frac{\Delta^3 y_0}{3} - \frac{\Delta^4 y_0}{4} + \cdots\right]$$

$$\left[\frac{d^2y}{dx^2}\right]_{\text{at } x=x_0} = \frac{1}{h^2}\left[\Delta^2 y_0 - \Delta^3 y_0 + \frac{11}{12}\Delta^4 y_0 + \cdots\right]$$

$$\left[\frac{d^3y}{dx^3}\right]_{at\ x=x_0} = \frac{1}{h^3}\left[\Delta^3 y_0 - \frac{3}{2}\Delta^4 y_0 + \cdots\right]$$

Newton's Backward difference formula to compute the derivatives

For Non-tabular values

$$\frac{dy}{dx} = \frac{1}{h}\left[\nabla y_n + \left(\frac{2p+1}{2}\right)\nabla^2 y_n + \left(\frac{3p^2+6p+2}{6}\right)\nabla^3 y_n + \left(\frac{2p^3+9p^2+11p+3}{12}\right)\nabla^4 y_n + \cdots\right]$$

$$\frac{d^2y}{dx^2} = \frac{1}{h^2}\left[\nabla^2 y_n + (p+1)\nabla^3 y_n + \frac{6p^2+18p+11}{12}\nabla^4 y_n + \cdots\right]$$

$$\frac{d^3y}{dx^3} = \frac{1}{h^3}\left[\nabla^3 y_n + \left(\frac{12p+18}{12}\right)\nabla^4 y_n + \cdots\right]$$

where $p = \frac{x-x_n}{h}$

For tabular values at $x = x_n$

$$\left[\frac{dy}{dx}\right]_{at\ x=x_n} = \frac{1}{h}\left[\nabla y_n + \frac{\nabla^2 y_n}{2} + \frac{\nabla^3 y_n}{3} + \frac{\nabla^4 y_n}{4} + \cdots\right]$$

$$\left[\frac{d^2y}{dx^2}\right]_{at\ x=x_n} = \frac{1}{h^2}\left[\nabla^2 y_n + \nabla^3 y_n + \frac{11}{12}\nabla^4 y_n + \cdots\right]$$

$$\left[\frac{d^3y}{dx^3}\right]_{at\ x=x_n} = \frac{1}{h^3}\left[\nabla^3 y_n + \frac{3}{2}\nabla^4 y_n + \cdots\right]$$

Example: 1

Find the first and second derivatives of the function tabulated below, at the point $x = 1.5$ and $x = 4$.

x	1.5	2.0	2.5	3.0	3.5	4.0
y	3.375	7.000	13.625	24.000	38.875	59.000

Solution:

Difference Table. Given that $h = 0.5$

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
1.5	$\boxed{3.375}$				
		$\boxed{3.625}$			
			$\boxed{3.000}$		
2.0	7.000			$\boxed{0.75}$	
		6.625			$\boxed{0}$
			3.750		
2.5	13.625			0.75	
		10.375			$\boxed{0}$
			4.500		
3.0	24.000			$\boxed{0.75}$	
		14.875			
			$\boxed{5.250}$		
3.5	38.875				
		$\boxed{20.125}$			
4.0	$\boxed{59.000}$				

Forward.

Forward.

backward.

Given $x = 1.5$ is a tabular value near the beginning of the given table.

Newton's Forward difference formula

$$\left[\frac{dy}{dx}\right]_{\text{at } x = x_0} = \frac{1}{h}\left[\Delta y_0 - \frac{\Delta^2 y_0}{2} + \frac{\Delta^3 y_0}{3} - \frac{\Delta^4 y_0}{4} + \cdots\right]$$

$$\left[\frac{dy}{dx}\right]_{\text{at } x = 1.5} = \frac{1}{0.5}\left[3.625 - \frac{3.000}{2} + \frac{0.75}{3}\right] = 4.75$$

$$\left[\frac{d^2y}{dx^2}\right]_{at\ x=x_0} = \frac{1}{h^2}\left[\Delta^2 y_0 - \Delta^3 y_0 + \frac{11}{12}\Delta^4 y_0 - \frac{5}{6}\Delta^5 y_0 + \cdots\right]$$

$$= \frac{1}{(0.5)^2}\left[3.000 - 0.75\right]$$

$$= \frac{1}{0.25}\left[3.000 - 0.75\right]$$

$$\left[\frac{d^2y}{dx^2}\right]_{at\ x=1.5} = 9.000$$

Newton's Backward difference formula

$x=4$ is at the end of the data

<u>First derivative</u>

$$\left[\frac{dy}{dx}\right]_{at\ x=x_n} = \frac{1}{h}\left[\nabla y_n + \frac{\nabla^2 y_n}{2} + \frac{\nabla^3 y_n}{3} + \cdots\right]$$

$$\left[\frac{dy}{dx}\right]_{at\ x=4} = \frac{1}{0.5}\left[20.125 + \frac{5.25}{2} + \frac{0.75}{3}\right] = 46$$

<u>Second derivative</u>

$$\left[\frac{d^2y}{dx^2}\right]_{at\ x=x_n} = \frac{1}{h^2}\left[\nabla^2 y_n + \nabla^3 y_n + \frac{11}{12}\nabla^4 y_n + \cdots\right]$$

$$= \frac{1}{(0.5)^2}\left[5.25 + 0.75 + \frac{11}{12}(0)\right]$$

$$= \frac{1}{0.25}\left[5.25 + 0.75\right]$$

$$\left[\frac{d^2y}{dx^2}\right]_{at\ x=4} = 24$$

Example: 2

Find the first and second derivatives of the function at the point $x=1.2$ from the following data

x	1	2	3	4	5
y	0	1	5	6	8

Solution:

Difference table

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
1	0				
		1		forward	
			3		
2	1			-6	
		4			10
			-3		
3	5			4	
		1			
			1		
4	6				
		2			
5	8				

At $x=1.2$, we can apply Newton's forward interpolation formula ~~for~~ to compute derivatives

First derivative

$$\frac{dy}{dx} = \frac{1}{h}\left[\Delta y_0 + \left(\frac{2p-1}{2}\right)\Delta^2 y_0 + \left(\frac{3p^2-6p+2}{6}\right)\Delta^3 y_0 + \left(\frac{2p^3-9p^2+11p-3}{6}\right)\Delta^4 y_0 + \cdots\right]$$

Here $h=1$, $x=1.2$. $x_0=1$

$$p = \frac{x-x_0}{h} = \frac{1.2-1}{1} = 0.2$$

$$\frac{dy}{dx} = \frac{1}{1}\left[1 + \frac{2(0.2)-1}{2}(3) + \frac{3(0.2)^2-6(0.2)+2}{6}(-6) + \frac{2(0.3)^3-9(0.2)^2+11(0.2)-3}{12}(10)\right]$$

$$= 1 + \frac{3}{2}[0.4-1] \frac{(0.12-1.2+2)(6)}{6} + \frac{10}{12}[0.016-0.36+2.2-3]$$

$$= 1 - 0.9 - 0.92 - 0.953$$

$$\frac{dy}{dx} = -1.773$$

second derivatives

$$\frac{d^2y}{dx^2} = \frac{1}{h^2}\left[\Delta^2 y_0 + (p-1)\Delta^3 y_0 + \frac{6p^2-18p+11}{12}\Delta^4 y_0 + \dots\right]$$

$$= \frac{1}{(1)^2}\left[3 + (0.2-1)(-6) + \frac{1}{12}[6(0.2)^2 - 18(0.2)+11] \cdot 10\right]$$

$$= 3 + 4.8 + 6.366$$

$$\frac{d^2y}{dx^2} = 14.17$$

Example: 3

Find $f'(x)$ and $f''(x)$ at $x = 2.9$ from the following data

x	1	1.5	2	2.5	3
y	27	106.75	324	783.75	1621

Solution:

Difference table

x	y	∇y	$\nabla^2 y$	$\nabla^3 y$	$\nabla^4 y$
1	27				
		79.75			
			137.5		
1.5	106.75			105	
		217.25			
			242.5		30
2	324			135	
		459.75			
			377.5		
2.5	783.75				
		837.25			
3	1621				

→ backward

At $x = 2.9$ is non tabular value.

we can apply Newton's backward difference formula

$h = 0.5 \quad p = \frac{x - x_n}{h} = \frac{2.9-3}{0.5} = \frac{-0.1}{0.5} = -0.2$

First derivative

$$f'(x) = \frac{dy}{dx} = \frac{1}{h}\left[\nabla y_n + \left(\frac{2p+1}{2}\right)\nabla^2 y_n + \left(\frac{3p^2+6p+2}{6}\right)\nabla^3 y_n + \left(\frac{2p^3+9p^2+11p+3}{12}\right)\nabla^4 y_n\right]$$

$$= \frac{1}{0.5}\left[837.5 + \frac{[2(-0.2)+1]}{2}(377.5) + \frac{[3(-0.2)^2+6(-0.2)+2]}{6}(135) + \frac{[2(-0.2)^3+9(-0.2)^2+11(-0.2)+3]}{12}(30)\right]$$

$$\boxed{f'(x) = 1936.68}$$

$$f''(x) = \frac{d^2y}{dx^2} = \frac{1}{h^2}\left[\nabla^2 y_n + (p+1)\nabla^3 y_n + \frac{6p^2+18p+11}{12}\nabla^4 y_n + \cdots\right]$$

$$= \frac{1}{(0.5)^2}\left[377.5 + (-0.2+1)\,135 + \frac{[6(-0.2)^2+18(-0.2)+11]}{12}(30)\right]$$

$$\boxed{f''(x) = 2018.4}$$

Example: 4

Find the value of sec 31° using the following data

$x°$	31	32	33	34
$\tan x$	0.6008	0.6249	0.6494	0.6745

Solution:

If $y = \tan x$

$$\frac{dy}{dx} = \sec^2 x$$

To find $\sec^2 31°$

Difference table

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
31	0.6008		forward	
		0.0241		
			0.0004	
				0.0002
32	0.6249			
		0.0245		
			0.0006	
33	0.6494			
		0.0251		
34	0.6745			

Here $h = 1° = 0.01745$ radians.

By Newtons forward interpolation formula.

$$\left[\frac{dy}{dx}\right]_{x=x_0} = \frac{1}{h}\left[\Delta y_0 - \frac{\Delta^2 y_0}{2} + \frac{\Delta^3 y_0}{3} - \frac{\Delta^4 y_0}{4} + \cdots\right]$$

$$= \frac{1}{0.01745}\left[0.0241 - \frac{1}{2}(0.0004) + \frac{1}{3}(0.0002)\right]$$

$$= \frac{1}{0.01745}\left[0.023967\right]$$

$$\sec^2 31° = 1.3732$$

$$\sec 31° = \sqrt{1.3732} = 1.1718 \quad \therefore \boxed{\sec 31° = 1.1718}$$

H.W

1. Find the first and second derivative of y at $x=15$ from the table below.

x	15	17	19	21	23	25
y	3.873	4.123	4.359	4.583	4.796	5.000

Ans

$\frac{dy}{dx} = 0.1291$, $\frac{d^2y}{dx^2} = -0.0046$.

2. Find the first two derivative of y at $x=54$ from the following table

x	50	51	52	53	54
y	3.6840	3.7084	3.7325	3.7563	3.7798

Ans

$\frac{dy}{dx} = 0.02335$ $\frac{d^2y}{dx^2} = -0.0003$

3. A rod is rotating in a plane. The angle θ (in radians) through which the rod has turned for various values of time t (seconds) are given below

t	0	0.2	0.4	0.6	0.8	1.0	1.2
θ	0	0.122	0.493	1.123	2.022	3.220	4.666

Ans Find the angular velocity and angular acceleration of the rod when $t = 0.6$ seconds.

Hint : angular velocity ⇒ first derivative $= \frac{d\theta}{dt}$

angular acceleration $= \frac{d^2\theta}{dt^2}$ (second derivative

$t=0.6$ we can apply Newtons backward diff formule.

Ans Angular velocity $= \frac{d\theta}{dt} = 3.81375$, Angular Acceleration $\Big\} = \frac{d^2\theta}{dt^2} = 6.7275$